

## FINITELY GENERATED POWERS OF PRIME IDEALS

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ABSTRACT. Let  $R$  be a commutative ring. If  $P$  is a maximal ideal of  $R$  whose a power is finitely generated then we prove that  $P$  is finitely generated if  $R$  is either locally coherent or arithmetical or a polynomial ring over a ring of global dimension  $\leq 2$ . And if  $P$  is a prime ideal of  $R$  whose a power is finitely generated then we show that  $P$  is finitely generated if  $R$  is either a reduced coherent ring or a polynomial ring over a reduced arithmetical ring. These results extend a theorem of Roitman, published in 2001, on prime ideals of coherent integral domains.

All rings are commutative and unitary. In this paper the following question is studied:

**question A:** Suppose that some power  $P^n$  of the prime ideal  $P$  of a ring  $R$  is finitely generated. Does it follows that  $P$  is finitely generated?

When  $P$  is maximal it is the *question 0.1* of [6], a paper by Gilmer, Heinzer and Roitman. The first author posed this question in [5, p.74]. In [6] some positive answers are given to the *question 0.1* (see [6, for instance, Theorem 1.24]), but also some negative answers (see [6, Example 3.2]). The authors proved a very interesting result ([6, Theorem 1.17]): a reduced ring  $R$  is Noetherian if each of its prime ideals has a finitely generated power. This *question 0.1* was recently studied in [10] by Mahdou and Zennayi, where some examples of rings with positive answers are given, but also some examples with negative responses. In [11] Roitman investigated the **question A**. In particular, he proved that  $P$  is finitely generated if  $R$  is a coherent integral domain ([11, Theorem 1.8]).

We first study *question 0.1* in Section 1. It is proven that  $P$  is finitely generated if  $R$  is either locally coherent or arithmetical. In Section 2 we investigate **question A** and extend the Roitman's result. We get a positive answer when  $R$  is a reduced ring which is either coherent or arithmetical. If  $R$  is not reduced, we obtain a positive answer for all prime ideals  $P$ , except if  $P$  is minimal and not maximal. In Section 3, by using Vasconcelos's results, we deduce that **question A** has also a positive response if  $R$  is a polynomial ring over either a reduced arithmetical ring or a ring of global dimension  $\leq 2$ . In Section 4, we consider rings of constant functions defined over a totally disconnected compact space  $X$  with values in a ring  $O$  for which a quotient space of  $\text{Spec } O$  has a unique point, and we examine when these rings give a positive answer to our questions. This allows us to provide some examples and counterexamples.

We denote respectively  $\text{Spec } R$ ,  $\text{Max } R$  and  $\text{Min } R$ , the space of prime ideals, maximal ideals and minimal prime ideals of  $R$ , with the Zariski topology. If  $A$  a

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subset of  $R$ , then we denote  $(0 : A)$  its annihilator and

$$V(A) = \{P \in \text{Spec } R \mid A \subseteq P\} \text{ and } D(A) = \text{Spec } R \setminus V(A).$$

### 1. POWERS OF MAXIMAL IDEALS

Recall that a ring  $R$  is **coherent** if each finitely generated ideal is finitely presented. It is well known that  $R$  is coherent if and only if  $(0 : r)$  and  $A \cap B$  are finitely generated for each  $r \in R$  and any two finitely generated ideals  $A$  and  $B$ .

**Theorem 1.1.** *Let  $R$  be a coherent ring. If  $P$  is a maximal ideal such that  $P^n$  is finitely generated for some integer  $n > 0$  then  $P$  is finitely generated too.*

*Proof.* First, suppose there exists an integer  $n > 0$  such that  $P^n = 0$ . So,  $R$  is local of maximal ideal  $P$ . We can choose  $n$  minimal. If  $n = 1$  then  $P$  is clearly finitely generated. Suppose  $n > 1$ . It follows that  $P^{n-1} \neq 0$ . So,  $P = (0 : r)$  for each  $0 \neq r \in P^{n-1}$ . Since  $R$  is coherent,  $P$  is finitely generated. Now, suppose that  $P^n$  is finitely generated for some integer  $n \geq 1$ . If  $R' = R/P^n$  and  $P' = P/P^n$  then  $R'$  is coherent and  $P'^n = 0$ . From above we deduce that  $P'$  is finitely generated. Hence  $P$  is finitely generated too.  $\square$

The following theorem can be proven by using [6, Lemma 1.8].

**Theorem 1.2.** *Let  $R$  be a ring. Suppose that  $R_P$  is coherent for each maximal ideal  $P$ . If  $P$  is a maximal ideal such that  $P^n$  is finitely generated for some integer  $n > 0$  then  $P$  is finitely generated too.*

*Proof.* Suppose that  $P^n$  is generated by  $\{x_1, \dots, x_k\}$ . Let  $L \neq P$  be a maximal ideal. Let  $s \in P \setminus L$ . Then  $s^n \in P^n \setminus L$ . It follows that  $s^n R_L = P^n R_L = P R_L = R_L$ . So, there exists  $i$ ,  $1 \leq i \leq k$  such that  $P R_L = x_i R_L$ . Since  $R_P$  is coherent,  $P R_P$  is finitely generated by Theorem 1.1. So, there exist  $y_1, \dots, y_m$  in  $P$  such that  $P R_P = y_1 R_P + \dots + y_m R_P$ . Let  $Q$  be the ideal generated by  $\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_m\}$ . Then  $Q \subseteq P$  and it is easy to check that  $Q R_L = P R_L$  for each maximal ideal  $L$ . Hence  $P = Q$  and  $P$  is finitely generated.  $\square$

A ring  $R$  is a **chain ring** if its lattice of ideals is totally ordered by inclusion, and  $R$  is **arithmetical** if  $R_P$  is a chain ring for each maximal ideal  $P$ .

**Theorem 1.3.** *Let  $R$  be an arithmetical ring. If  $P$  is a maximal ideal such that  $P^n$  is finitely generated for some integer  $n > 0$  then  $P$  is finitely generated too.*

*Proof.* First, assume that  $R$  is local. Let  $P$  be its maximal ideal. Suppose that  $P$  is not finitely generated and let  $r \in P$ . Since  $P \neq Rr$  there exists  $a \in P \setminus Rr$ . So,  $r = ab$  with  $b \in P$ . It follows that  $P^2 = P$  and  $P^n = P$  for each integer  $n > 0$ . So,  $P^n$  is not finitely generated for each integer  $n > 0$ . Now, we do as in the proof of Theorem 1.2 to complete the demonstration.  $\square$

**Remark 1.4.** There exist arithmetical rings which are not coherent. In [10] several other examples of non-coherent rings which satisfy the conclusion of the previous theorem are given.

Let  $R$  be a ring. For a polynomial  $f \in R[X]$ , denote by  $c(f)$  (the content of  $f$ ) the ideal of  $R$  generated by the coefficients of  $f$ . We say that  $R$  is **Gaussian** if  $c(fg) = c(f)c(g)$  for any two polynomials  $f$  and  $g$  in  $R[X]$  (see [12]). A ring  $R$  is

said to be a **fqp-ring** if each finitely generated ideal  $I$  is projective over  $R/(0 : I)$  (see [1, Definition 2.1 and Lemma 2.2]).

By [1, Theorem 2.3] each arithmetical ring is a fqp-ring and each fqp-ring is Gaussian, but the converses do not hold. The following examples show that Theorem 1.3 cannot be extended to the class of fqp-rings and the one of Gaussian rings.

**Example 1.5.** *Let  $R$  be a local ring and  $P$  its maximal ideal. Assume that  $P^2 = 0$ . Then it is easy to see that  $R$  is a fqp-ring. But  $P$  is possibly not finitely generated.*

**Example 1.6.** *Let  $A$  be a valuation domain (a chain domain),  $M$  its maximal ideal generated by  $m$  and  $E$  a vector space over  $A/M$ . Let  $R = \left\{ \begin{pmatrix} a & e \\ 0 & a \end{pmatrix} \mid a \in A, e \in E \right\}$  be the trivial ring extension of  $A$  by  $E$ . We easily check that  $R$  is a local fqp-ring. Let  $P$  be its maximal ideal. Then  $P^2$  is generated by  $\begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix}$ . But, if  $E$  is of infinite dimension over  $A/M$  then  $P$  is not finitely generated over  $R$  (see also [10, Theorem 2.3(iv)a]).*

## 2. POWERS OF PRIME IDEALS

By [11, Theorem 1.8], if  $R$  is a coherent integral domain then each prime ideal whose a power is finitely generated is finitely generated too. The following example shows that this result does not extend to any coherent ring.

**Example 2.1.** *Let  $D$  be a valuation domain. Suppose there exists a non-zero prime ideal  $L'$  which is not maximal. Moreover assume that  $L' \neq L'^2$  and let  $d \in L' \setminus L'^2$ . If  $R = D/Dd$  and  $L = L'/Dd$ , then  $R$  is a coherent ring,  $L$  is not finitely generated and  $L^2 = 0$ .*

**Remark 2.2.** *Let  $R$  be an arithmetical ring. In the previous example we use the fact that each non-zero prime ideal  $L$  which is not maximal is not finitely generated. In Theorem 2.9 we shall prove that  $L^n$  is not finitely generated for each integer  $n > 0$  if  $L$  is not minimal.*

In the sequel let  $\Phi = \text{Max } R \cup (\text{Spec } R \setminus \text{Min } R)$  for any ring  $R$ .

The proof of the following theorem is similar to that of [11, Theorem 1.8].

**Theorem 2.3.** *Let  $R$  be a coherent ring. Then, for any  $P \in \Phi$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

*Proof.* Let  $P \in \Phi$  such that  $P^k$  is finitely generated for some integer  $k > 0$ . By Theorem 1.1 we may assume that  $P$  is not maximal. So, there exists a minimal prime ideal  $P'$  such that  $P' \subset P$ . It follows that  $P^n \neq 0$  for each integer  $n > 0$ . By [11, Lemma 1.7] there exist an integer  $n > 1$  such that  $P^n$  is finitely generated and  $a \in P^{n-1} \setminus P^{(n)}$  where  $P^{(n)}$  is the inverse image of  $P^n R_P$  by the natural map  $R \rightarrow R_P$ . This implies that  $aP = aR \cap P^n$ . We may assume that  $a \notin P'$ , else, we replace  $a$  with  $a+b$  where  $b \in P^n \setminus P'$ . Since  $R$  is coherent,  $aP$  and  $(0 : a)$  are finitely generated. From  $a \notin P'$  we deduce  $(0 : a) \subseteq P' \subset P$ , whence  $P \cap (0 : a) = (0 : a)$ . Hence  $P$  is finitely generated.  $\square$

**Corollary 2.4.** *Let  $R$  be a reduced coherent ring. Then, for any prime ideal  $P$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

*Proof.* Let  $P$  be a prime ideal of  $R$  such that  $P^n$  is finitely generated for some integer  $n > 1$ . We may assume that  $P \neq 0$  and by Theorem 2.3 that  $P$  is minimal. So,  $P^n \neq 0$ . It is easy to check that  $(0 : P) = (0 : P^n)$  because  $R$  is reduced.

Since  $R$  is coherent, it follows that  $(0 : P)$  is finitely generated. On the other hand, since  $P^n$  is finitely generated, there exists  $t \in (0 : P^n) \setminus P$ . This implies that  $P = (0 : (0 : P))$ . We conclude that  $P$  is finitely generated.  $\square$

An exact sequence of  $R$ -modules  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  is **pure** if it remains exact when tensoring it with any  $R$ -module. Then, we say that  $F$  is a **pure** submodule of  $E$ . The following proposition is well known.

**Proposition 2.5.** [4, Proposition 2.4] *Let  $A$  be an ideal of a ring  $R$ . The following conditions are equivalent:*

- (1)  $A$  is a pure ideal of  $R$ ;
- (2) for each finite family  $(a_i)_{1 \leq i \leq n}$  of elements of  $A$  there exists  $t \in A$  such that  $a_i = a_i t$ ,  $\forall i$ ,  $1 \leq i \leq n$ ;
- (3) for all  $a \in A$  there exists  $b \in A$  such that  $a = ab$  (so,  $A = A^2$ );
- (4)  $R/A$  is a flat  $R$ -module.

Moreover:

- if  $A$  is finitely generated, then  $A$  is pure if and only if it is generated by an idempotent;
- if  $A$  is pure, then  $R/A = S^{-1}R$  where  $S = 1 + A$ .

If  $R$  is a ring, we consider on  $\text{Spec } R$  the equivalence relation  $\mathcal{R}$  defined by  $L\mathcal{R}L'$  if there exists a finite sequence of prime ideals  $(L_k)_{1 \leq k \leq n}$  such that  $L = L_1$ ,  $L' = L_n$  and  $\forall k$ ,  $1 \leq k \leq (n-1)$ , either  $L_k \subseteq L_{k+1}$  or  $L_k \supseteq L_{k+1}$ . We denote by  $\text{pSpec } R$  the quotient space of  $\text{Spec } R$  modulo  $\mathcal{R}$  and by  $\lambda : \text{Spec } R \rightarrow \text{pSpec } R$  the natural map. The quasi-compactness of  $\text{Spec } R$  implies the one of  $\text{pSpec } R$ , but generally  $\text{pSpec } R$  is not  $T_1$ : see [8, Propositions 6.2 and 6.3].

**Lemma 2.6.** [4, Lemma 2.5]. *Let  $R$  be a ring and let  $C$  a closed subset of  $\text{Spec } R$ . Then  $C$  is the inverse image of a closed subset of  $\text{pSpec } R$  by  $\lambda$  if and only if  $C = V(A)$  where  $A$  is a pure ideal. Moreover, in this case,  $A = \bigcap_{P \in C} 0_P$ .*

In the sequel, for each  $x \in \text{pSpec } R$  we denote by  $A(x)$  the unique pure ideal which verifies  $\overline{\{x\}} = \lambda(V(A(x)))$ , where  $\overline{\{x\}}$  is the closure of  $\{x\}$  in  $\text{pSpec } R$ .

**Theorem 2.7.** *Let  $R$  be a ring. Assume that  $R/A(x)$  is coherent for each  $x \in \text{pSpec } R$ . Then, for any  $P \in \Phi$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

*Proof.* Let  $P \in \Phi$  and  $I = A(\lambda(P))$ . Suppose that  $P^n$  is generated by  $\{x_1, \dots, x_k\}$ . Let  $L$  be a maximal ideal such that  $I \not\subseteq L$ . As in the proof of Theorem 1.2 we show that  $PR_L = x_i R_L$  for some integer  $i$ ,  $1 \leq i \leq k$ . By Theorem 2.3  $P/I$  is finitely generated over  $R/I$ . So, there exist  $y_1, \dots, y_m$  in  $P$  such that  $(y_1 + I, \dots, y_m + I)$  generate  $P/I$ . Let  $Q$  be the ideal generated by  $\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_m\}$ . Then  $Q \subseteq P$  and it is easy to check that  $QR_L = PR_L$  for each maximal ideal  $L$ . Hence  $P = Q$  and  $P$  is finitely generated.  $\square$

From Corollary 2.4, Theorem 2.7 and the fact that  $R/I$  is a localization of  $R$  for each pure ideal  $I$ , we deduce the following.

**Corollary 2.8.** *Let  $R$  be a reduced ring. Assume that  $R/A(x)$  is coherent for each  $x \in \text{pSpec } R$ . Then, for any prime ideal  $P$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

**Theorem 2.9.** *Let  $R$  be an arithmetical ring. Then, for any  $P \in \Phi$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

*Proof.* Let  $P$  be a prime ideal. By Theorem 1.3 we may assume that  $P$  is not maximal. Let  $M$  be a maximal ideal containing  $P$ . If  $P$  is not minimal then  $P^n R_M$  contains strictly the minimal prime ideal of  $R_M$  for each integer  $n > 0$ . So,  $P^n R_M \neq 0$  for each integer  $n > 0$ . On the other hand, since  $R_M$  is a chain ring it is easy to check that  $PR_M = MPR_M$ . It follows that  $P^n R_M = MP^n R_M$  for each integer  $n > 0$ . By Nakayama Lemma we deduce that  $P^n R_M$  is not finitely generated over  $R_M$ . Hence,  $P^n$  is not finitely generated for each integer  $n > 0$ .  $\square$

**Remark 2.10.** Example 2.1 shows that the assumption " $P \in \Phi$ " cannot be omitted in some previous results. However, if each minimal prime ideal which is not maximal is idempotent then the conclusions hold for each prime ideal  $P$ .

### 3. PF-RINGS

Now, we consider the rings  $R$  for which each prime ideal contains a unique minimal prime ideal. So, the restriction  $\lambda'$  of  $\lambda$  to  $\text{Min } R$  is bijective. In this case, for each minimal prime ideal  $L$  we put  $A(L) = A(\lambda(L))$ . By [3, Proposition IV.1]  $\text{pSpec } R$  is Hausdorff and  $\lambda'$  is a homeomorphism if and only if  $\text{Min } R$  is compact. We deduce the following from Lemma 2.6.

**Proposition 3.1.** *Let  $R$  be a ring. Assume that each prime ideal contains a unique minimal prime ideal. Then, for each minimal prime ideal  $L$ ,  $V(L) = V(A(L))$ . Moreover, if  $R$  is reduced then  $A(L) = L$ .*

*Proof.* If  $R$  is reduced, then for each  $P \in V(L)$ ,  $LR_P = 0$  whence  $L = 0_P$ .  $\square$

We say that a ring  $R$  is a **pf-ring** if one of the following equivalent conditions holds:

- (1)  $R_P$  is an integral domain for each maximal ideal  $P$ ;
- (2) each principal ideal of  $R$  is flat;
- (3) each cyclic submodule of a flat  $R$ -module is flat.

Moreover, if  $R$  is a pf-ring then each prime ideal  $P$  contains a unique minimal prime ideal  $P'$  and  $A(P') = P'$  by Proposition 3.1.

So, from the previous section and the fact each minimal prime ideal of a pf-ring is idempotent, we deduce the following three results. Let us observe that each prime ideal of an arithmetical ring  $R$  contains a unique minimal prime ideal because  $R_P$  is a chain ring for each maximal ideal  $P$ .

**Corollary 3.2.** *Let  $R$  be a coherent pf-ring. Then each prime ideal whose a power is finitely generated is finitely generated too.*

**Corollary 3.3.** *Let  $R$  be a pf-ring. Assume that  $R/L$  is coherent for each minimal prime ideal  $L$ . Then each prime ideal whose a power is finitely generated is finitely generated too.*

**Corollary 3.4.** *Let  $R$  be a reduced arithmetical ring. Then each prime ideal whose a power is finitely generated is finitely generated too.*

The following three corollaries allows us to give some examples of pf-ring satisfying the conclusion of Corollary 3.3. Let  $n$  be an integer  $\geq 0$  and  $G$  a module over a ring  $R$ . We say that  $\text{pd } G \leq n$  if  $\text{Ext}_R^{n+1}(G, H) = 0$  for each  $R$ -module  $H$ .

**Corollary 3.5.** *Let  $R$  be a coherent ring. Assume that each finitely generated ideal  $I$  satisfies  $\text{pd } I < \infty$ . Then each prime ideal whose a power is finitely generated is finitely generated too.*

*Proof.* By, either [2, Théorème A] or [7, Corollary 6.2.4],  $R_P$  is an integral domain for each maximal ideal  $P$ . So,  $R$  is a pf-ring.  $\square$

**Corollary 3.6.** *Let  $A$  be a ring and  $X = \{X_\lambda\}_{\lambda \in \Lambda}$  a set of indeterminates. Consider the polynomial ring  $R = A[X]$ . Assume that  $A$  is reduced and arithmetical. Then each prime ideal of  $R$  whose a power is finitely generated is finitely generated too.*

*Proof.* Let  $P$  be a maximal ideal of  $R$  and  $P' = P \cap A$ . Thus  $R_P$  is a localization of  $A_{P'}[X]$ . Since  $A_{P'}$  is a valuation domain,  $R_P$  is an integral domain. So,  $R$  is a pf-ring. Now, let  $P$  be a minimal prime ideal of  $R$  and  $L$  be a minimal prime ideal of  $A$  contained in  $P \cap A$ . We put  $A' = A/L$  and  $R' = A'[X]$ . So,  $A'$  is an arithmetical domain (a Prüfer domain). By [13, Proposition 8.2(b)]  $R'$  is coherent. Since  $R/P$  is flat over  $R$  and  $R'$ ,  $R/P$  is a localization of  $R'$ . Hence  $R/P$  is coherent. We conclude by Corollary 3.3.  $\square$

Let  $n$  be an integer  $\geq 0$ . We say that a ring  $R$  is of global dimension  $\leq n$  if  $\text{pd } G \leq n$  for each  $R$ -module  $G$ .

**Corollary 3.7.** *Let  $A$  be a ring and  $X = \{X_\lambda\}_{\lambda \in \Lambda}$  a set of indeterminates. Consider the polynomial ring  $R = A[X]$ . Assume that  $A$  is of global dimension  $\leq 2$ . Then each prime ideal of  $R$  whose a power is finitely generated is finitely generated too.*

*Proof.* Let  $P$  be a maximal ideal of  $R$  and  $P' = P \cap A$ . Thus  $R_P$  is a localization of  $A_{P'}[X]$ . Since  $A_{P'}$  is an integral domain by [9, Lemme 2],  $R_P$  is an integral domain. So,  $A$  and  $R$  are pf-rings. By [9, Proposition 2]  $A/L$  is coherent for each minimal prime ideal  $L$ . Now, we conclude as in the proof of the previous corollary, by using [13, Theorem 8.5].  $\square$

#### 4. MINIMAL PRIME IDEALS

A topological space is called **totally disconnected** if each of its connected components contains only one point. Every Hausdorff topological space  $X$  with a base of clopen (closed and open) neighbourhoods is totally disconnected and the converse holds if  $X$  is compact (see [14, Lemma 29.6]).

**Proposition 4.1.** *Let  $R$  be a ring. Let  $P$  be a minimal prime ideal such that  $P^n$  is finitely generated for some integer  $n > 0$ . Then  $P$  is an isolated point of  $\text{Min } R$ .*

*Proof.* It is well known that  $V(I) \cap \text{Min } R$  is a clopen subset of  $\text{Min } R$  for any finitely generated ideal  $I$ . Since  $V(P^n) \cap \text{Min } R = \{P\}$ ,  $P$  is an isolated point of  $\text{Min } R$  if  $P^n$  is finitely generated.  $\square$

From Theorems 2.7 and 2.9 and Proposition 4.1 we deduce the following theorem.

**Theorem 4.2.** *Let  $R$  be a ring. Assume that  $R$  is either coherent or arithmetical and  $\text{Min } R$  contains no isolated point. Then, each prime ideal whose a power is finitely generated is finitely generated too.*

**Proposition 4.3.** *Let  $R$  be a ring for which each prime ideal contains only one minimal prime ideal. Let  $P$  be a minimal prime ideal such that  $P^n$  is finitely generated for some integer  $n > 0$ . Then  $\lambda(P)$  is an isolated point of  $\text{pSpec } R$ .*

*Proof.* Let  $P$  be a minimal prime ideal and  $A = A(\lambda(P))$ . Clearly  $\lambda(P) = V(P) = V(A)$ . Since  $R/A$  is flat, we have  $A^2 = A$ . From  $A \subseteq P$  we deduce that  $A \subseteq P^2$ . It follows that  $A \subseteq P^n$  for each integer  $n > 0$ . Suppose that  $P^n$  is finitely generated for some integer  $n > 0$ . Since  $P/A$  is the nilradical of  $R/A$ ,  $P^m = A$  for some integer  $m \geq n$ . We deduce that  $P^m = Re$  for some idempotent  $e$  of  $R$  by Proposition 2.5. It follows that  $\lambda(P) = V(P^m) = D(1 - e)$ . Hence  $\lambda(P)$  is an isolated point of  $\text{pSpec } R$ .  $\square$

**Proposition 4.4.** *Let  $X$  be a totally disconnected compact space, let  $O$  be a ring with a unique point in  $\text{pSpec } O$ . Let  $R$  be the ring of all locally constant maps from  $X$  into  $O$ . Then,  $\text{pSpec } R$  is homeomorphic to  $X$  and  $R/A(z) \cong O$  for each  $z \in \text{pSpec } R$ .*

*Proof.* If  $U$  is a clopen subset of  $X$  then there exists an idempotent  $e_U$  defined by  $e_U(x) = 1$  if  $x \in U$  and  $e_U(x) = 0$  else. Let  $x \in X$  and  $\phi_x : R \rightarrow O$  be the map defined by  $\phi_x(r) = r(x)$  for every  $r \in R$ . Clearly  $\phi_x$  is a ring homomorphism, and since  $R$  contains all the constant maps,  $\phi_x$  is surjective. Let  $x \in X$ ,  $r \in \ker(\phi_x)$  and  $U = \{y \in X \mid r(y) \neq 0\}$ . Then  $U$  is a clopen subset. It is easy to check that  $e_U \in \ker(\phi_x)$  and  $r = re_U$ . Since  $\ker(\phi_x)$  is generated by idempotents,  $R/\ker(\phi_x)$  is flat over  $R$ . For each  $x \in X$ , let  $\Pi(x)$  be the image of  $\text{Spec } O$  by  $\lambda \circ \phi_x^a$  where  $\phi_x^a : \text{Spec } O \rightarrow \text{Spec } R$  is the continuous map induced by  $\phi_x$ . We shall prove that  $\Pi : X \rightarrow \text{pSpec } R$  is a homeomorphism. Clearly,  $V(\ker(\phi_x)) \subseteq \Pi(x)$ . Conversely, let  $P \in \Pi(x)$ . Then there exists  $L \in V(\ker(\phi_x))$  such that  $PRL$ . We may assume that  $L \subseteq P$  or  $P \subseteq L$ . The first case is obvious. For the second case let  $e$  an idempotent of  $\ker(\phi_x)$ . Then,  $e \in L$ ,  $(1 - e) \notin L$ ,  $(1 - e) \notin P$  and  $e \in P$ . We conclude that  $V(\ker(\phi_x)) = \Pi(x)$  because  $\ker(\phi_x)$  is generated by its idempotents. Let  $x, y \in X$ ,  $x \neq y$ . By using the fact there exists a clopen subset  $U$  of  $X$  such that  $x \in U$  and  $y \notin U$  then  $e_U \in \ker(\phi_y)$  and  $(1 - e_U) \in \ker(\phi_x)$ . So,  $\ker(\phi_x) + \ker(\phi_y) = R$ , whence  $\Pi$  is injective. By way of contradiction suppose there exists a prime ideal  $P$  of  $R$  such that  $\ker(\phi_x) \not\subseteq P$  for each  $x \in X$ . There exists an idempotent  $e'_x \in \ker(\phi_x) \setminus P$  whence  $e_x = (1 - e'_x) \in P \setminus \ker(\phi_x)$ . Let  $V_x$  be the clopen subset associated with  $e_x$ . Clearly  $X = \cup_{x \in X} V_x$ . Since  $X$  is compact, a finite subfamily  $(V_{x_i})_{1 \leq i \leq n}$  covers  $X$ . We put  $U_1 = W_1 = V_{x_1}$ , and for  $k = 2, \dots, n$ ,  $W_k = \cup_{i=1}^k V_{x_i}$  and  $U_k = W_k \setminus W_{k-1}$ . Then  $U_k$  is clopen for each  $k = 1, \dots, n$ . For  $i = 1, \dots, n$  let  $\epsilon_i \in R$  be the idempotent associated with  $U_i$ . Since  $U_i \subseteq V_{x_i}$ , we have  $\epsilon_i = e_{x_i} \epsilon_i$ . So,  $\epsilon_i \in P$  for  $i = 1, \dots, n$ . It is easy to see that  $1 = \sum_{i=1}^n \epsilon_i$ . We get  $1 \in P$ . This is false. Hence  $\Pi$  is bijective. We easily check that  $x \in U$ , where  $U$  is a clopen subset of  $X$ , if and only if  $\Pi(x) \subseteq D(e_U)$ . Since  $A(\Pi(x)) = \ker(\phi_x)$  is generated by its idempotents,  $\text{pSpec } R$  has a base of clopen neighbourhoods. We conclude that  $\Pi$  is a homeomorphism.  $\square$

From Corollary 2.8 we deduce the following proposition.

**Proposition 4.5.** *Let  $R$  be the ring defined in Proposition 4.4. Assume that  $O$  is a reduced coherent ring. Then, for any prime ideal  $P$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

**Proposition 4.6.** *Let  $R$  be the ring defined in Proposition 4.4. Assume that  $O$  has a unique minimal prime ideal  $M$ . Then, every prime ideal of  $R$  contains only one minimal prime ideal and  $\text{Min } R$  is compact. If  $M = 0$  then  $R$  is a pp-ring, i.e. each principal ideal is projective.*

*Proof.* If  $P$  is a prime ideal of  $R$  then there exists a unique  $x \in X$  such that  $P \in \Pi(x)$ . So,  $\phi_x^a(M)$  is the only minimal prime ideal contained in  $P$ .

Assume that  $M = 0$ . Let  $r \in R$ ,  $e = e_U$  where  $U$  is the clopen subset of  $X$  defined by  $U = \{x \in X \mid r(x) \neq 0\}$ . We easily check that the map  $Re \rightarrow Rr$  induced by the multiplication by  $r$  is an isomorphism. This proves that  $R$  is a pp-ring.

Let  $R'$  be the ring obtained like  $R$  by replacing  $O$  with  $O/M$ . It is easy to see that  $R' \cong R/N$  where  $N$  is the nilradical of  $R$ . So,  $\text{Min } R$  and  $\text{Min } R'$  are homeomorphic. Since  $R'$  is a pp-ring,  $\text{Min } R$  is compact by [13, Proposition 1.13].  $\square$

From Theorems 2.7 and 2.9 and Propositions 4.3 and 4.6 we deduce the following corollary.

**Corollary 4.7.** *Let  $R$  be the ring defined in Proposition 4.4. Suppose that  $O$  has a unique minimal prime ideal  $M$ . Assume that  $O$  is either coherent or arithmetical and that one of the following conditions holds:*

- (1)  $M$  is either idempotent or finitely generated;
- (2)  $X$  contains no isolated point.

*Then, for any prime ideal  $P$ ,  $P$  is finitely generated if  $P^n$  is finitely generated for some integer  $n > 0$ .*

**Example 4.8.** *Let  $R$  be the ring defined in Proposition 4.4. Assume that:*

- $O$  is either coherent or arithmetical, with a unique minimal prime ideal  $M$ ;
- $M$  is not finitely generated and  $M^k = 0$  for some integer  $k > 1$  (for example,  $O$  is the ring  $R$  defined in Example 2.1);
- $X$  contains no isolated points (for example the Cantor set, see [14, Section 30]).

*Then the property "for each prime ideal  $P$ ,  $P^n$  is finitely generated for some integer  $n > 0$  implies  $P$  is finitely generated" is satisfied by  $R$ , but not by  $R/A(L)$  for each minimal prime ideal  $L$ .*

From Theorems 1.2 and 1.3 and Proposition 4.3 we deduce the following corollary.

**Corollary 4.9.** *Let  $R$  be the ring defined in Proposition 4.4. Assume that  $O$  is local with maximal ideal  $M$ . Then each prime ideal of  $R$  is contained in a unique maximal ideal, and for each maximal ideal  $P$ ,  $R_P \cong O$ . Moreover, if one of the following conditions holds:*

- (1)  $O$  is coherent;
- (2)  $O$  is a chain ring;
- (3)  $X$  contains no isolated point and  $M$  is the sole prime ideal of  $O$ .

*then, for each maximal ideal  $P$ ,  $P^n$  finitely generated for some integer  $n > 0$  implies  $P$  is finitely generated.*

**Example 4.10.** *Let  $R$  be the ring defined in Proposition 4.4. Assume that  $M$  is the sole prime ideal of  $O$ ,  $M$  is not finitely generated,  $M^k = 0$  for some integer  $k > 1$  and  $X$  contains no isolated points. Then the property "for each maximal ideal*



*$P$ ,  $P^n$  is finitely generated for some integer  $n > 0$  implies  $P$  is finitely generated” is satisfied by  $R$ , but not by  $R_L$  for each maximal ideal  $L$ .*

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